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Stable power laws in variable economies; Lotka-Volterra implies Pareto-Zipf

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Abstract. In recent years we have found that logistic systems of the Generalized Lotka-Volterra type (GLV) describing statistical systems of auto-catalytic elements posses power law distributions of the Pareto-Zipf type. In particular, when applied to economic systems, GLV leads to power laws in the relative individual wealth distribution and in market returns. These power laws and their exponent α are invariant to arbitrary variations in the total wealth of the system and to other endogenously and exogenously induced variations.

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1 Logistic equations, GLV and power laws

The logistic equation (1)

$$\mathrm{d}w/\mathrm{d}t = Aw - Bw^2 \tag{1}$$

has been used for more than 100 years to describe various biological, demographic and economic systems. Montroll [2] claims that "almost all the social phenomena, except in their relatively brief abnormal times obey the logistic growth". Lotka [3] and Volterra [4] interpreted w as the size of an animal/plant population, Aw as the aggregated effects of birth and natural death, and $-Bw^2$ as the effects of the competition for limited resources.

In economics, Aoki [5] interpreted w as the total product demand in a market. The linear term Aw models the emergence of new products that are proportional to the present size of the market. The nonlinear term $-Bw^2$ expresses the fact that the products have to compete with one another within a finite total potential market.

Solomon and Levy [6] have suggested that w can represent the total capital within a financial system. In this interpretation, the first term represents the average returns that the system offers, while the term $-Bw^2$ represents the effects of competition and other growth limiting factors.

An apparently universal (and until recently un-related) property, spanning a wide range of disciplines from linguistics to economics and to biology is the presence of scale-invariant probability distributions [7]. This property was initially observed by Pareto [8] more than 100 years ago in the individual wealth distribution: for each economy, the fraction $P(w)$ of people owning a wealth w is proportional to a power of w:

$$P(w) \sim w^{-1-\alpha}. \tag{2}$$

The presence of Pareto power laws equation (2) in dynamical systems with random multiplicative dynamics has been known experimentally for many years. It occurs in a variety of fields: the frequency of words in texts [9], economic growth [10], cities populations [11], wealth distribution [12], renewal stochastic processes [13], "$1/f$ noise" phenomena in engineering [14] and physiology [15] etc.

It was shown [6,16] that systems of the type equation (1), when studied at the level of microscopic agents rather than in the aggregate form equation (1), lead to power law distributions of the form equation (2). These Generalized Lotka-Volterra (GLV) systems [17,18] treat each component of the system individually while taking into account their non-linear interactions. GLV explains not only the ubiquitous emergence of the power laws in many fields but also their stability in generic systems with non-stationary dynamics and arbitrarily varying total size [19,20]. In particular, GLV explains measured values of the exponent of the Pareto wealth distribution in terms of the social and biological constraints on the economy [21].

One can therefore say that the careful reconsideration of the system equation (1) has led to the solution of a puzzle that is over 100 years old by an equation that is also over 100 years old.

In the next section we introduce the GLV model and its various interpretations. In Section 3 we show how GLV reduces to a set of decoupled stationary linear stochastic

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differential equations with constant coefficients. In Section 4 we derive analytically the Pareto law for the relative wealth distribution in the GLV model.

2 Definition and interpretation of GLV

We describe the dynamics of the GLV system in the discrete time formulation to avoid ambiguities related to the continuum (Ito *vs.* Stratonovich) formulation. The time evolution of the system from time t to time $t+\tau$ is given by the recursive equation [6,17–19]:

$$w_i(t+\tau) - w_i(t) = r_i(t)w_i(t) + aw(t) - c(w.,t)w_i(t) \quad (3)$$

where

• $w(t)$ is the average $\langle w_i(t)\rangle$ over all i's at time t.

• the functions a and $c(w.,t)$ are of order τ in order to insure a meaningful "continuum limit" $\tau \to 0$.

• the notation $c(w.,t)$, means that $c(w_1, w_2, ..., w_N, t)$ can depend in an arbitrary (un-symmetric, time dependent, way) on each of the $w_j(t)$'s.

• $r_i(t)$'s are random numbers (of order unity) distributed with the same probability distribution (independent of i) with a square standard deviation D of order τ

$$\langle r_i(t)^2\rangle = D. \quad (4)$$

• One can absorb the average $\langle r_i(t)\rangle$ into the arbitrary function $c(w.,t)$ and assume without loss in generality

$$\langle r_i(t)\rangle = 0. \quad (5)$$

The system equation (3) admits a few practical interpretations.

If one considers $w_i(t)$ as the individual wealth of the agent i, then:

• the random multiplicative factor $r_i(t)$ represents the random part of the returns that its capital $w_i(t)$ produces during the time between t and $t+\tau$.

• The coefficient a expresses the auto-catalytic property of wealth at the social level, *i.e.* it represents the wealth that individuals receive as members of the society in subsidies, services and social benefits. This is the reason it is proportional to the average wealth. This term prevents, as we shall show, the individual wealth falling below a certain minimum fraction of the average. The exact mechanism by which this happens (subsidies, minimal insurance or wage, elimination of the weak and their substitution by the more fit) is not, at this level of description, important.

• The coefficient $c(w.,t)$ controls the overall growth of the wealth in the system. It represents external limiting factors: finite amount of resources and money in the economy, technological inventions, wars, disasters etc. It also includes internal market effects: competition between investors, adverse influence of bids on prices (such as when large investors sell assets to realize their profits and cause thereby prices/ profits to fall). This term has the effect of limiting the growth of $w(t)$ to values sustainable for current conditions and resources.

$c(w.,t)$ parametrizes the general state of the economy. Time periods during which $-c(w.,t)$ is large and positive correspond to boom periods during which the wealth is on average increasing. Periods during which $-c(w.,t)$ is negative correspond to recessions, when typically the investments lead to negative or small returns. The surprising fact (proven in Sect. 4) is that as long as the term $c(w.,t)$ and the distribution of the $r_i(t)$'s are common for all the i's, the Pareto power law equation (2) holds and its exponent is independent on $c(w.,t)$. This an important finding since the i-independence corresponds to the famous market efficiency property in financial markets.

A different interpretation of GLV equation (3) may consider the market as a set of companies $i = 1, ..., N$ whose shares are traded at variable prices $w_i(t)$. The price of each stock $w_i(t)$ is proportional to the capitalization of the corresponding company i (the total wealth of all the market shares of the company). In this case,

• $r_i(t)$ represents fluctuations in the market worth of the company. For a fixed total number of market shares, $r_i(t)$ also measures relative changes in individual share prices. These changes are typically fractions of the nominal share price (measured in percents or in points).

• aw represents correlation between the worth of each company w_i and the market index $w(t)$.

• The non-linear term, which in this interpretation has usually the particular form $-c(t)w(t)w_i(t)$ represents competition between the companies for finite amounts of money in the market (and limits their worth). Time variations in global resources may lead to lower or higher values of $c(t)$. These in turn lead to increases or decreases in the total (or average) wealth $w(t)$.

Yet another interpretation of the GLV equation (3) is in the context of the investors herding behavior:

• $w_i(t)$ is the number of traders adopting a similar investment policy or position (they comprise "herd" i).

• one assumes that the sizes of these sets vary autocatalytically according to the random factor $r_i(t)$. This can be justified by the fact that the visibility and social connections of a herd are proportional to its size.

• aw represents the diffusion of traders between the herds.

• The nonlinear term $c(w.,t)$ represents the general status of popularity of the stock market as a whole. This term also includes the competition between various herds in attracting individual traders as members.

3 Reducing GLV to simple stochastic differential equations

Many properties of the GLV nonlinear system of coupled differential equations with time-dependent (and variable-dependent) coefficients equation (3) can be studied analytically. To do this, let us first take the average in both members of equation (3) and get (assuming that in the $N \to \infty$ limit the random fluctuations cancel according 2.3 (see however [16,20,22])):

$$w(t+\tau) - w(t) = aw(t) - c(w.,t)w(t). \quad (6)$$

Equation (6) reduces in the continuum limit to a differential equation of the form equation (1). Therefore, at the aggregate level, the system described by the equation (3) represents the same system as equation (1) (with the identifications $A = a/\tau$ and $Bw = c(w., t)/\tau)$). However, the "microscopic representation" equation (3) allows one to uncover properties that would be impossible to guess from contemplating equation (1). Introducing the new variable

$$x_i(t) = w_i(t)/w(t) \tag{7}$$

and applying the chain rule for differentials $\mathrm{d}x_i = x_i(t+\tau) - x_i(t)$, $\mathrm{d}w_i = w_i(t+\tau) - w_i(t)$ and $\mathrm{d}w = w(t+\tau) - w(t)$:

$$\mathrm{d}x_i(t) = \frac{\mathrm{d}w_i}{w(t)} - \frac{w_i(t)}{w(t)} \frac{\mathrm{d}w}{w} \,. \tag{8}$$

Equation (3) becomes (considering Eqs. (6) and (7))

$$\mathrm{d}x_i(t) = r_i(t)x_i(t) + a - c(w., t)x_i(t) - x_i(t)[a - c(w., t)]. \tag{9}$$

At this stage a very crucial cancellation takes place: the nonlinear, time dependent function $c(w., t)$ that coupled the equations of the system disappears. Consequently the system splits into a set of independent linear stochastic differential equations with constant coefficients.

$$\mathrm{d}x_i(t) = [r_i(t) - a]x_i + a \tag{10}$$

Note that, so far, we have not assumed that the system of w_i's is in a steady state, yet we have been able to show that the stochastic dynamics of the relative individual wealths x_i reduces to a set of identical decoupled linear equations equation (10) which are independent on $c(w., t)$. The combination D/a representing the ratio between the fluctuations of the speculative income and the additive socially insured income is the only parameter influencing the relative wealth dynamics.

In particular, even in the presence of large arbitrary time variations of $c(w., t)$ and $w(t)$, if a/D is constant, the relative wealth will eventually reach a time independent distribution that we compute analytically in the next section. The approach of this asymptotic distribution by the x_i's is governed by the equations equation (10) and therefore is itself independent of the global non-stationary dynamics induced by $c(w., t)$ on $w(t)$.

In fact, equation (10) holds for a wider range of models:

$$w_i(t+\tau) - w_i(t) = r_i(t)w_i(t) + a_i \sum_j b_j w_j(t) - c(w., t)w_i(t) \tag{11}$$

where a_i and b_i are arbitrary positive coefficients. This corresponds to a social security system which assembles a budget proportional to b_j of the wealth of each individual j and distributes to each individual i a fraction a_i of this budget.

By multiplying each equation (11) (for each i) by b_i and summing, one gets (see however [19,22]):

$$u(t+\tau) - u(t) = au(t) - c(w., t)u(t) \tag{12}$$

where we used the notation $u(t) = \sum_j b_j w_j(t)$ and $a(t) = \sum_j a_j b_j(t)$. Now perform the change of variables:

$$x_i(t) = w_i(t)/u(t) \tag{13}$$

and use the differential chain rule

$$\mathrm{d}x_i = \frac{\mathrm{d}w_i}{u} - \frac{w_i \mathrm{d}u}{u^2} \,. \tag{14}$$

From equations (11, 12) and (13) we obtain

$$\mathrm{d}x_i = r_i(t)x_i(t) + a - c(w., t)x_i(t) - x_i(t)[a - c(w., t)] \tag{15}$$

Again the nonlinear time dependent arbitrary function $c(w_1, w_2, ..., w_N, t)$ that couple the equations of the system equation (11) cancels leaving only a set of uncoupled time independent linear stochastic differential equations with constant coefficients equation (10).

4 Stable Pareto distribution in GLV

Equation (10) leads to a stationary probability distribution that can be computed analytically. Note that the convergence to the stationary distribution of the relative wealths $x_i(t) = w_i(t)/u(t)$ is guaranteed by equation (10) even if the system defined by equation (11) has a very non-stationary dynamics.

Now compute the stationary distribution corresponding to the generic stochastic differential dynamics:

$$x(t+\tau) - x(t) = \varepsilon(t)g(x(t)) + f(x(t)). \tag{16}$$

We work explicitly with discrete time steps τ to avoid the ambiguities related to Ito *vs.* Stratonovich interpretations of the continuous stochastic differential equations [23,24]. The result equation (24) will then be applied to the particular case equation (10).

Without loss of generality, we can assume $\langle \varepsilon(t) \rangle = 0$ since the non-random part of $\varepsilon(t)$ can be absorbed in a redefinition of $f \to f + \langle \varepsilon(t) \rangle g$.

In order for the noise $\varepsilon(t)$ to be relevant as one takes "the continuum limit" $\tau \to 0$ we assume the square standard deviation:

$$D = \langle \varepsilon(t)^2 \rangle \tag{17}$$

to be of order τ.

As a consequence, we have to keep terms of order $\varepsilon(t)^2$ and thus also occasionally terms of second order in the differential $\mathrm{d}x = x(t+\tau) - x(t)$.

For a meaningful "continuum limit", the function $f(x)$ is taken to be of order τ while $g(x)$ is of order 1.

In order to find the asymptotic probability distribution corresponding to the dynamics equation (16) we will perform an appropriate change of variables $y(t) = y(x(t))$ and obtain a Langevin process with constant (unit) coefficient for the random term:

$$y(t+\tau) - y(t) = \varepsilon(t) + j(y(t)). \tag{18}$$

This yields the (Maxwell-Boltzmann) stationary distribution [25] which is the exponential of the integral of the "drift force" j normalized to the "thermal term" $D/2$:

$$P(y)\mathrm{d}y = \exp\left[\frac{2}{D}\int^{y} j(z)\mathrm{d}z\right]\mathrm{d}y. \tag{19}$$

The time evolution equation for the new variable $y(t)$ is obtained from the one for $x(t)$, equation (16), using the chain differential rule (in order to keep the terms of order D we expand up to second order in $\mathrm{d}x$):

$$y(x(t+\tau)) - y(x(t)) = \mathrm{d}y = \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x + 1/2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}(\mathrm{d}x)^2 + \text{etc.}$$
$$= \frac{\mathrm{d}y}{\mathrm{d}x}[\varepsilon(t)g(x(t)) + f(x(t)] + \frac{D}{2}g^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \text{etc.} \tag{20}$$

where etc. on the r.h.s. denotes terms that vanish faster than τ in the limit $\tau \to 0$.

Obviously, in order to bring equation (16) to the form equation (18) using equation (20), one needs to make the particular change of variables:

$$\mathrm{d}y = \frac{1}{g}\mathrm{d}x. \tag{21}$$

With this change, equation (20) becomes:

$$y(t+\tau) - y(t) = \varepsilon(t) + \frac{f(x(y))}{g(x(y))} - \frac{D}{2}\frac{\mathrm{d}g}{\mathrm{d}x}\,\cdot \tag{22}$$

According to equation (19), this leads to the asymptotic probability distribution:

$$P(y)\mathrm{d}y = \exp\left[\frac{2}{D}\left(\int^{y}\left(\frac{f(x(z))}{g(x(z))} - \frac{D}{2}\frac{\mathrm{d}g(x(z))}{\mathrm{d}x}\right)\mathrm{d}z\right)\right]\mathrm{d}y. \tag{23}$$

$$P(x) = \exp\left[\frac{2}{D}\int^{x}\frac{f(v)}{g^2(v)}\mathrm{d}v\right]\frac{\mathrm{d}x}{g(x)^2}\,\cdot \tag{24}$$

In order to find the stationary distribution of $x_i(t) = w_i(t)/w(t)$ corresponding to the dynamics equation (10), all one has to do is to apply equation (24) to the particular case:

$$f(x) = a(1-x)$$

and

$$g(x) = x.$$

Thus according to equation (24):

$$P(x)\mathrm{d}x = \exp\left[\frac{2}{D}\int^{x}\frac{a-v}{v^2}\mathrm{d}v\right]x^{-2}\mathrm{d}x.$$

After performing the integrals one obtains:

$$P(x)\mathrm{d}x = x^{-1-\alpha}\exp[-2a/(xD)] \tag{25}$$

with

$$\alpha = 1 + 2a/D. \tag{26}$$

This result has important implications for all systems such as those described the Sections 1 and 2. In effect, even during very unstable conditions that lead to non-stationary global dynamics, ecologies, economies, stock markets, physiological systems, communication networks, social systems, continue to be characterized by stable power laws with time invariant exponents. These dimensionless exponents depend only on ratios such as:

– that between the additive income and the volatility of the mutiplicative speculative incomes: $\alpha = 1 + 2a/D$.

– or the ratio between the effective minimal wealth $w_{\min}$ and the average wealth w:

$$\alpha = \frac{1}{[1 - w_{\min}(t)/w(t)]}.$$

For instance if L is the average number of dependents on the average wealth/income owner, then the average wealth is about L times the minimal wealth (necessary to keep alive one person). Consequently one predicts [20,21] $\alpha \sim \frac{L}{L-1} \sim 1.5$ which is in good agreement with the experimental data.

Moreover, it can be shown [16–21] that the dynamics equation (11) implies that the scaling exponents of the market returns distribution $\mathrm{d}w/w$ and of the Pareto Law $P(w)$ are equal. This prediction is borne out by the data [7,28].

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